Generation and verification of high-dimensional entanglement from coupled-cavity arrays

Dimitris G. Angelakis

Centre for Quantum Computation, Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

Sougato Bose

Department of Physics and Astronomy, University College London, Gower Street, London WC1E 6BT, UK

Received May 12, 2006; revised August 18, 2006; accepted September 26, 2006; posted October 11, 2006 (Doc. ID 70536); published January 26, 2007

We show how coupled cavities can be used to produce high-dimensional entangled states of electromagnetic fields. We also show how such an entangled state can be verified by mapping the entangled fields to atoms or quantum dots in the defects. We propose this as a source of high dimensional entangled states on demand and suggest ways to implement it using coupled defects in photonic crystals or coupled toroidal microcavities.

OCIS codes: 270.1670, 020.1670.

1. INTRODUCTION

Shared entanglement between distant parties is a very important concept for quantum information processing (QIP).[1,2] Entangled optical fields are highly suited for this purpose, and light is very effective for long-distance communication. Usually either parametric downconversion producing polarization entangled photons[3,4] or continuous variable (CV) entanglement using interference of squeezed states[5] is used as sources of entanglement in QIP. While CV entanglement can be high dimensional, discrete variable entanglement so far has mostly been that between two two-state systems (or a single ebit[3,4]). A notable exception is the higher dimensional entangled states recently studied by Howell *et al.*,[6] which is equivalent to the entanglement between two three-level systems. This is related to earlier theoretical work by Drummond[7] and Reid *et al.*[8] In this paper, we show how entanglement equivalent to that between two three-level systems and two four-level systems can be created using coupled cavities for electromagnetic fields. We also show how dopants such as atoms or quantum dots in these cavities can be used to both prepare the initial state needed for generating this entanglement and probe the final entangled state. We conclude by discussing the possibility of implementing these ideas using either defects in photonic bandgap materials or toroidal microcavities coupled via tapered optical fibers. This is motivated by the recent availability of high-Q cavities near the strong coupling regime that in addition can be strongly and efficiently coupled to each other.[9–14] If one seeks very preliminary applications of such systems in quantum information, then the proposal of our paper comes across as one of the simplest possible of such applications.

2. SYSTEM

We consider the system depicted in Fig. 1 with two coupled cavities. Let a and b be the field operators for the photonic modes in each defect. The Hamiltonian describing the hopping of photons from one cavity to another in such a system of two neighboring cavities is

$$H = a^{\dagger}b + ab^{\dagger}. \tag{1}$$

This coupling is strongest when modes a and b are resonant to each other. As we will assume this to be the case whenever the Hamiltonian H is used, we ignore the energy terms $\propto a^{\dagger}a + b^{\dagger}b$ of the two modes.

We also assume that each cavity is doped with a multilevel system (atom or quantum dot) of level configuration, as shown in the lower part of Fig. 1. The dopant in the defect with mode a is labeled A and the the dopant in the defect with mode b is labeled B. They are assumed to have a set of lower Zeeman levels (labeled $|g_i\rangle\, i=0,\ldots,N$) and a set of higher Zeeman levels (labeled $|e_i\rangle\, i=0,\ldots,N-1$). The case for $N=3$ with atoms as dopants is shown in Fig. 1. In the context of cavity QED it has been shown[15] that adiabatic passage enables the following back and forth mapping of internal states of such atoms to Fock states in cavities in which they are trapped:

$$|g_r\rangle|n\rangle \leftrightarrow |g_{r-k}\rangle|n+k\rangle. \tag{2}$$

Using these dopants, the cavities are first prepared in Fock states $|k\rangle_a|k\rangle_b$ [one simply starts the dopants in the state $|g_k\rangle_A|g_k\rangle_B$—the cavities empty, i.e., in state $|0\rangle_a|0\rangle_b$—and performs the mapping of Eq. (4)]. We assume that during this mapping the influence of H can be

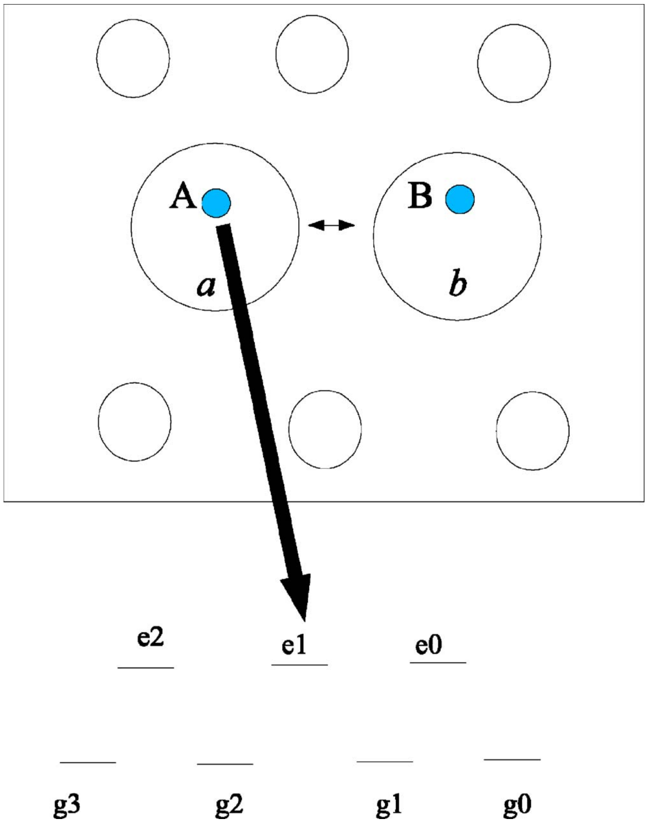

Fig. 1. (Color online) Coupled doped cavities in PBG crystals. The cavities support modes a and b of light of the same frequency. These modes are coupled owing to hopping of photons between the cavities. The cavities are doped with dopants A and B, whose level configuration is shown in the bottom of the diagram. Mappings of the form $|g_k\rangle|0\rangle \leftrightarrow |g_0\rangle|k\rangle$ are possible through the application of external fields using adiabatic passage techniques (Ref. 15).

neglected using external fields at detuning the two defects from resonance.

3. EVOLUTION AND ENTANGLEMENT

We first consider the case of $k=1$, or in terms of operators, the initial state to be $a^\dagger b^\dagger|0\rangle$. The time evolution of the system when the Hamiltonian H acts on it can be found out by the following evolution of the operators:

$$a^\dagger \to a^\dagger \cos t + i b^\dagger \sin t,$$

$$b^\dagger \to b^\dagger \cos t + i a^\dagger \sin t. \tag{3}$$

The above leads to the following state of the two modes in time:

$$|\psi(t)\rangle_{ab} = \cos t|1\rangle_a|1\rangle_b + i\frac{\sin t}{\sqrt{2}}(|2\rangle_a|0\rangle_b + |0\rangle_a|2\rangle_b). \tag{4}$$

Thus at time $t=\tan^{-1}\sqrt{2}$, the state $(1/\sqrt{3})(|1\rangle_a|1\rangle_b + |2\rangle_a|0\rangle_b + |0\rangle_a|2\rangle_b)$ is obtained. This is equivalent to a maximally entangled state of two spin-1 systems. However, entanglement of two spin-1 systems using optical modes has already been seen in Ref. 6 albeit in a polarization and photon number combined setting. Thus the entangled state obtained with the initial state $k=1$ is not a significant advance over available entangled states.

We now examine the case when we start with initial state $k=2$. Equation (4) can still be used to compute the time evolution of the initial state $a^{\dagger 2} b^{\dagger 2}|0\rangle$, but we have also computed the evolution fully numerically to check that this procedure gives the right answer. The numerical results give the following general state of the two modes as a function of time:

$$|\phi(t)\rangle_{ab} = c_{22}|2\rangle_a|2\rangle_b + c_{31}|3\rangle_a|1\rangle_b + c_{13}|1\rangle_a|3\rangle_b + c_{40}|4\rangle_a|0\rangle_b + c_{04}|0\rangle_a|4\rangle_b, \tag{5}$$

where the amplitudes c_{ij} vary with time, as given in Fig. 2. According to Eq. (4) the state is given by

$$|\phi(t)\rangle_{ab} = (1/2)\left[2\left(1 - \frac{3\sin^2 2t}{2}\right)|2\rangle_a|2\rangle_b - \frac{\sqrt{6}\sin^2 2t}{2}(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b) + i\sqrt{6}\sin 2t \cos 2t(|1\rangle_a|3\rangle_b + |3\rangle_a|1\rangle_b)\right], \tag{6}$$

which is in complete agreement with the numerics. From Fig. 2 note a special point at which c_{22} completely vanishes. At this point the state has a particularly simple form (obtainable by setting $\sin^2 2t = 2/3$) given by

$$|\phi_E\rangle_{ab} = \frac{1}{\sqrt{3}}(|3\rangle_a|1\rangle_b + |1\rangle_a|3\rangle_b) + \frac{1}{\sqrt{6}}|4\rangle_a|0\rangle_b + c_{04}|0\rangle_a|4\rangle_b. \tag{7}$$

The Schmidt form of the above state immediately suggests that it is equivalent to an entangled state to two spin-3/2 (or two four-level) systems. But how entangled is it? One can compute the entanglement of this state from the von Neumann entropy of the reduced density matrix of either of the modes given by $S=-Tr\rho_a \log \rho_a = -\Sigma_i |c_i|^2 \log |c_i|^2$. It is found to be 1.9183 ebits. Given that two four-level systems can have 2 ebits of entanglement

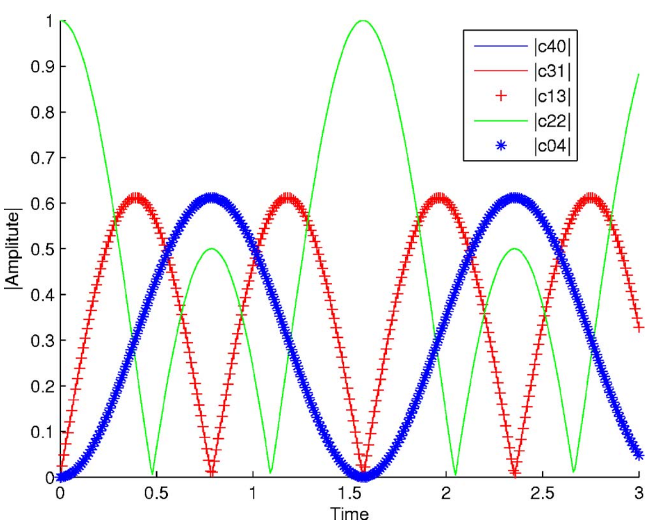

Fig. 2. (Color online) Time dependence of the amplitudes assuming initially the system was at $|2\rangle_a|2\rangle_b$.

at most, the state $|\phi_E\rangle_{ab}$ is indeed a highly entangled state of such systems.

4. VERIFICATION

To detect the generated entangled states such as $|\phi_E\rangle_{ab}$, we again resort to the multilevel dopants that dope the cavities and use the mapping of Eq. (4). It is assumed that the mapping takes place at a much faster time scale than the coupling between the cavities, and thus H has no effect on this mapping. The mapping transforms the entangled state of the modes a and b to the following entangled states of dopants:

$$c_{22}|2\rangle_a|2\rangle_b + c_{31}|3\rangle_a|1\rangle_b + c_{13}|1\rangle_a|3\rangle_b + c_{40}|4\rangle_a|0\rangle_b + c_{04}|0\rangle_a|4\rangle_b \rightarrow \tag{8}$$

$$c_{22}|g_2\rangle_A|g_2\rangle_B + c_{31}|g_3\rangle_A|g_1\rangle_B + c_{13}|g_1\rangle_A|g_3\rangle_B + c_{40}|g_4\rangle_A|g_0\rangle_B + c_{04}|g_0\rangle_A|g_4\rangle_B. \tag{9}$$

The amount of entanglement between the two dopants can be subsequently evaluated by performing a Bell inequality test for a bipartite system of high dimensionality such as the ones proposed in Refs. 16 and 17. In our case, as the required measurements will be performed on atomic states by external lasers, shelving, and fluorescence, rather than on photons with much higher detection efficiencies, high accuracy is expected in detecting the high dimension of entanglement.

5. IMPLEMENTATION

We now discuss how the proposed entanglement generation can be relevant to certain physical systems. A system of coupled high-Q cavities is essentially required. The cavity-field decay rate κ is required to be much smaller than the coupling strength A between two defects (A would be the term multiplying $a^{\dagger}b+ab^{\dagger}$ in the Hamiltonian). This is required so that there is virtually no decay from the cavities during the evolution of the entangled state from the initial Fock states, and the assumption of treating this evolution entirely unitarily is valid. One recent implementation, namely, toroidal microcavities coupled by tapered fibers, currently have A a few times (about 2.5 times) κ. Here we have estimated A from the difference in the quality factor of a fiber loaded cavity and the intrinsic quality factor of the same cavity. We note that as the intrinsic Q factors of these cavities are expected to be increased nearly fourfold in the near future to reach 4×10^8, A is expected to be about 1 order of magnitude greater than κ, which should suffice for our analysis to be valid.[9] In these cavities, the atom-field coupling strength g, which sets the time scale of the mapping of states from the cavity fields to the atomic levels, is about 40 times greater than the current values to κ (and consequently about 1 order of magnitude greater than A). Thus the mapping of the entangled electromagnetic field states to those of the atoms can be accomplished in a time scale much faster than both the evolution of these states due to the coupling of the cavities and any decay of these states due to cavity decay. The fact that atoms with the requisite level configuration for our scheme (many Zeeman levels) is possible and can be used for cavity QED is stated in Ref. 15. The Zeeman levels have virtually infinite lifetime in comparison to the time-scale of the dynamics of our problem. In the case of toroidal microcavities in chips, the technology of trapping atoms is also currently available,[18] which can be made to interact with the evanescent cavity fields.

An alternative interesting scenario for the implementation of our protocol can be photonic crystals (PCs). Photonic crystals or photonic bandgap materials (PBGs) are ordered artificial dielectrics.[19–21] The periodicity in the refractive index allows them to tailor the flow of light and, in some cases completely inhibit it, creating bandgaps in three dimensions. The introduction of line defects in an otherwise ordered crystal gives rise to allowed modes within the bandgap, where guided light can propagate with very low losses. Point defects can also be introduced leading to localized modes of light acting as very high Q cavities,[21] whereas chains with large number of defects have been predicted to allow efficient waveguiding through a hopping mechanism and fabricated experimentally.[22,23]

PBG defect cavities can also be doped with discrete level systems (atoms or a quantum dot)[10] that are a requirement for our protocol. More importantly, the dopant can be controlled by external lasers and can couple strongly to the optical modes in the high-Q cavities. In this strong coupling regime, the cavity field-atom coupling g can be as strong as 10^5 MHz and about 3.9 times larger than κ, which suffices for our purpose.[9] The coupling A between PBG cavities can be as high as terahertz (in Ref. 14), and the coupling between traveling mode waveguides and a defect cavity is given—one should be able to make the coupling between two cavities to be of the same order of magnitude). We require a lower A (so that $g>A$, for the validity of our analysis), and this should be possible by separating the defect cavities by a longer distance in the crystal. Moreover, one should be able to probe the internal states of the dopants by the usual shelving techniques. In other recent studies,[24–26] another set of feasible parameters for coupled cavities in PCs in the strong coupling ($g>\kappa$) regime as required here, as well as $g>A$ is discussed. In these studies the possibility to dope the PBG cavities with two-level atoms or other qubits (such as nitrogen vacancy centers in diamond) is examined.

6. CONCLUSION

We show that high-dimensional entanglement using electromagnetic fields (equivalent to the maximal entanglement of two spin-1 systems and highly entangled state of two spin-$\frac{3}{2}$ systems) can be prepared using coupled cavities. We have provided two examples of physical implementations of such a scheme, one using coupled toroidal microcavities and the other using coupled defect cavities in PBG crystals. If these states can be leaked out of the cavities and transmitted to long distances, one can have a shared high-dimensional entanglement between distant parties. We note here that Q switching individual PBG cavities and coupling them to a waveguide has already been discussed in Ref. 26, and similar ideas should be applicable to toroidal cavities as well. Atoms or dopants in the cavities both create the initial states required for entanglement generation, as well as serve as systems on which the entangled state can be mapped for detection. To keep the complexity of the scheme to a modest level (both in terms of initial state preparation and entangled state verification), we have just used initial states $|1\rangle_a|1\rangle_b$ and $|2\rangle_a|2\rangle_b$. However, in the future one could study the cases of starting with higher Fock states in the cavities. In the context of higher Fock states in cavities and entanglement generation from them, one is automatically also testing the bosonic nature of the photons, in much the same sense as their bunching in a 50–50 beam splitter tests bosonic statistics. This is an additional motivation for this work, as statistical experiments have yet to be performed with multiphoton Fock states incident on beam splitters (the coupling between the cavities in our scheme effectively simulates a beam splitter).

ACKNOWLEDGMENTS

We acknowledge the hospitality of the Kavli Institute for the Theoretical Physics in Santa Barbara, California, USA where part of the discussions leading to this work took place during the program on Topological Phases and Quantum Computation 2006. Also, the hospitality of the quantum information group at National University of Singapore, where part of this work was done during mutual research visits, is acknowledged. This work was supported in part by the Quantum Information Processing Interdisciplinary Research Collaboration (GR/S82176/01), the European Union, FP6-FET Integrated Project Scalable Quantum Computing with Light and Atoms, and the Cambridge-Massachusetts Institute of Technology Institute.

D. G. Angelakis's e-mail address is dimitris.angelakis@qubit.org.

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