Fractional Quantum Hall State in Coupled Cavities

Jaeyoon Cho,[1,2] Dimitris G. Angelakis,[2,3] and Sougato Bose[1,2]

[1]Department of Physics and Astronomy, University College London, Gower Street, London WC1E 6BT, United Kingdom
[2]Centre for Quantum Technologies, National University of Singapore, 2 Science Drive 3, Singapore 117542
[3]Science Department, Technical University of Crete, Chania, Crete, Greece, 73100

(Received 11 July 2008; published 11 December 2008)

We propose a scheme to realize the fractional quantum Hall system with atoms confined in a two-dimensional array of coupled cavities. Our scheme is based on simple optical manipulation of atomic internal states and intercavity hopping of virtually excited photons. It is shown that, as well as the fractional quantum Hall system, any system of hard-core bosons on a lattice in the presence of an arbitrary Abelian vector potential can be simulated solely by controlling the phases of constantly applied lasers. The scheme, for the first time, exploits the core advantage of coupled cavity simulations, namely, the individual addressability of the components, and also brings the gauge potential into such simulations as well as the simple optical creation of particles.

DOI: 10.1103/PhysRevLett.101.246809 PACS numbers: 73.43.−f, 03.67.Lx, 42.50.Pq

The achievement of trapping ultracold atomic gases in a strongly correlated regime has prompted an interest in mimicking various condensed matter systems, thereby allowing one to tackle such complex systems in unprecedented ways [1]. A major class of simulable systems, distinct from the Hubbard model and spin systems, is that in a gauge potential, such as the fractional quantum Hall (FQH) system. The FQH effect arises when a two-dimensional (2D) electron gas is in the presence of a strong perpendicular magnetic field at a low temperature. The Hall resistance of such a system exhibits plateaus when the Landau filling factor ν takes simple rational values [2]. The FQH effect at fundamental filling factors $\nu = 1/m$ for odd integers m (even integers for bosons) is accounted for by Laughlin's trial wave function (in the symmetric gauge) [3]

$$\Psi_m(\{z_j\}) = e^{-(1/4)\sum_j |z_j|^2} \prod_{j<k} (z_j - z_k)^m, \tag{1}$$

where $z_j = x_j + iy_j$ is the 2D position of the jth electron in unit of the magnetic length $l_B \equiv \sqrt{\hbar/eB}$, with B being the magnetic field. The elementary excitation of this state is a quasihole (quasiparticle), which has a fractional charge $+e/m$ $(-e/m)$ and obeys the anyonic statistics [4]. To simulate such a system in trapped atoms, a major challenge is to create an artificial magnetic field as the atoms in consideration have no real charge. This is done with considerable difficulties, for instance, by rapidly rotating the harmonic trap [5], by exploiting electromagnetically induced transparency [6], or by modulating the optical lattice potential [7,8]. Additionally, FQH systems are also simulable in Josephson junction arrays [9].

Recently, coupled cavity arrays (CCAs) [10–12] have emerged as a fascinating alternative for simulating quantum many-body phenomena, supported by diverse technologies, such as microwave stripline resonators, photonic crystal defects, microtoroidal cavity arrays, and so forth [13–15]. CCAs have complementary advantages over optical lattices, such as arbitrary many-body geometries and individual addressability [16]. Recently, theoretical works have shown that the Mott-superfluid phase transition of polaritons [10,11] and the Heisenberg spin chains [12] can be realized in CCAs. These works, however, relied only on globally addressing lasers and thus could not highlight the key advantage of CCAs, namely, the individual addressability, in the sense that already they can be done similarly or better in optical lattices [17]. Moreover, simulating altogether distinct classes of systems such as those of itinerant particles in a gauge potential still remains open, and this will be especially arresting if the particles themselves can be created by a purely optical means. In this Letter, we bring the Abelian gauge potential into the realm of many-body simulations using CCAs. We achieve this by actively exploiting the individual addressability, which eventually enables great versatility which has not been attainable in optical lattices.

As a concrete example, we introduce a way of simulating FQH systems in CCAs. To be more specific, we consider a FQH system of bosonic particles confined in a 2D square lattice of spacing a in the presence of a perpendicular and uniform artificial magnetic field B. Noninteracting free bosons in a single Bloch band are described by the Hamiltonian

$$H_0 = -t \sum_{\langle j,k \rangle} c_j^\dagger c_k \exp\left(-i \frac{2\pi}{\Phi_0} \int_j^k \mathbf{A}(\mathbf{r}) \cdot dl\right), \tag{2}$$

where c_j denotes a boson annihilation operator at site j and $\Phi_0 \equiv h/e$ is the magnetic flux quantum. The summation runs over nearest neighbor pairs. If we take a Landau gauge, this Hamiltonian is written as

$$H_0 = -t \sum_{p,q} (c_{p+1,q}^\dagger c_{p,q} e^{-i2\pi\alpha q} + c_{p,q+1}^\dagger c_{p,q} + \text{H.c.}), \tag{3}$$

where the positions of lattice sites are represented by $a(p\hat{x} + q\hat{y})$, with a being the spacing of the lattice. Here

$\alpha \equiv Ba^2/\Phi_0$, the number of magnetic flux quanta through a lattice cell, plays a crucial role in characterizing the energy spectrum, whose self-similar structure is known as the Hofstadter butterfly [18]. In addition to this noninteracting Hamiltonian, we consider a hard-core interaction between bosons, which limits the number of particles that can occupy one site to a maximum of one. In this limit, if we also take a continuum limit $\alpha \ll 1$, the Laughlin state (1) is a very accurate variational ground state, where the filling factor ν corresponds to the ratio of the number of bosons to the number of magnetic flux quanta.

In order to realize the above situation, we consider a two-dimensional array of coupled cavities, each confining a single atom with two ground levels, which will be representing an $s = \frac{1}{2}$ spin. First, we notify that, aside from the additional phases, the Hamiltonian (3) in the hard-core limit corresponds to that of an $s = \frac{1}{2}$ spin lattice system with XX interaction, where the creation-annihilation operation of the zero and one boson states is analogous to the spin flip operation of the spin-down and -up states. This natural realization of the hard-core limit is contrary to the case of optical lattices, wherein it is achieved in the limit of strong on-site repulsion [7,8]. Moreover, as will be seen later, every aspect of the system is optically controlled: Bosons are created by simple optical pulses, and the phases in the Hamiltonian are adjusted simply by controlling the phases of applied lasers. This optical control of the system would greatly simplify the experiments, compared to the previous schemes involving mechanical modulations of the system. Although in this work we mainly consider the FQH systems, another great advantage is that, unlike the previous schemes for optical lattices, any Abelian vector potential on a lattice can be also simulated simply by adjusting the laser phases in accordance with the formula (2). The creation of a quasiexcitation, which is achieved by adiabatically inserting a flux quantum through an infinitely thin magnetic solenoid piercing the 2D plane [3], again reduces to the matter of adiabatically changing the laser phases accordingly. It can be moved along the lattice cells by modulating the laser phases, which would be useful for testing the fractional statistics.

Schemes for realizing the spin exchange Hamiltonian in an array of coupled cavities have been established in recent papers [12]. In these schemes, the spin exchange is mediated by intercavity hopping of virtually excited cavity photons. An important difference in the present case is that the spin exchanges are associated with phase changes depending on their locations and directions, which obviously cannot be excluded by local phase transformations. For this reason, we introduce an asymmetry in the 2D geometry of coupled cavities, as shown schematically in Fig. 1, where two orthogonal cavity modes along the $\hat{x}$ and $\hat{y}$ directions have different resonant frequencies. Realizing this geometry would be viable in several promising models for coupled cavities, such as photonic band gap microcavities [13] and superconducting microwave cavities [14]. We assume the frequency difference between the two modes is much larger than the atom-cavity coupling rates. In this way, either direction of the spin exchange can be accessed individually by choosing the laser frequency. We note, however, that the above asymmetry is, in fact, not essential for our purpose. For example, an array of microtoroidal cavities [15], in which case realizing such a geometry is nontrivial, can be also used by involving more lasers in the scheme. We discuss this point later.

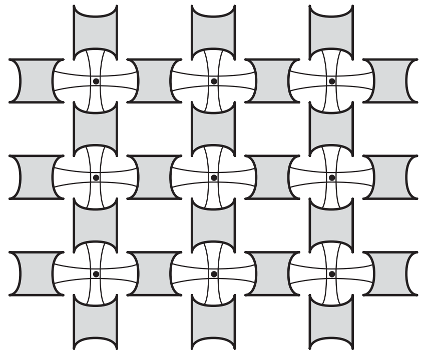

FIG. 1. Schematic representation of a two-dimensional array of coupled cavities. Each atom is confined in the intersection of two orthogonal cavity modes, which are adjusted to have different resonant frequencies.

We consider a simple atomic level and transition scheme as shown in Fig. 2. The atom has two ground levels $|0\rangle$ and $|1\rangle$ and an excited level $|e\rangle$. There are two cavity modes along the $\hat{x}$ and $\hat{y}$ directions, whose annihilation operators are denoted by a^X and a^Y, respectively. The atom interacts with these cavity modes with coupling rates g^X and g^Y and with detunings Δ^X and Δ^Y, respectively. Two classical fields with (complex) Rabi frequencies $\Omega^X e^{-i\theta^X}$ and $\Omega^Y e^{-i\theta^Y}$ are applied, respectively, as in the figure. In the rotating frame, the Hamiltonian reads

$$H = \sum_{\mu=X,Y} \sum_{j=(p,q)} [g^\mu e^{-i\Delta^\mu t} a_j^\mu (|e\rangle\langle 0|)_j + \Omega^\mu e^{-i\theta_j^\mu} e^{-i\Delta^\mu t} (|e\rangle\langle 1|)_j + \text{H.c.}] - \sum_{p,q} (J^X a_{p+1,q}^{X\dagger} a_{p,q}^X + J^Y a_{p,q+1}^{Y\dagger} a_{p,q}^Y + \text{H.c.}), \quad (4)$$

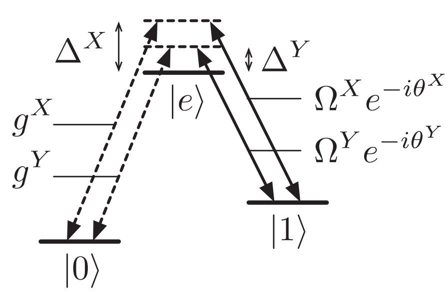

FIG. 2. Involved atomic levels and transitions. There are two independent Raman transitions mediated by an excited level $|e\rangle$ between two ground levels $|0\rangle$ and $|1\rangle$, represented by superscripts X and Y, respectively.

where J^X (J^Y) denotes the intercavity hopping rate of the photon along the $\hat{x}$ ($\hat{y}$) direction and the subscript (p, q) represents the cavity site. As mentioned above, we assume $\Delta^X - \Delta^Y \gg g^X, g^Y$ and also assume $\Delta^\mu \gg g^\mu \gg \Omega^\mu, J^\mu$. This requires the strong atom-cavity coupling in that $g^\mu \gg J^\mu$. In this regime, the atomic excitation is suppressed, and adiabatic elimination leads to an effective Hamiltonian

$$H = \sum_{\mu=X,Y} \sum_{j=(p,q)} [\delta^\mu a_j^{\mu\dagger} a_j^\mu (|0\rangle\langle 0|)_j + \omega^\mu (e^{i\theta_j^\mu} a_j^\mu \sigma_j^+ + \text{H.c.})] - \sum_{p,q} (J^X a_{p+1,q}^{X\dagger} a_{p,q}^X + J^Y a_{p,q+1}^{Y\dagger} a_{p,q}^Y + \text{H.c.}), \quad (5)$$

where $\delta^\mu = (g^\mu)^2/\Delta^\mu$, $\omega^\mu = g^\mu \Omega^\mu/\Delta^\mu$, and $\sigma^+ = |1\rangle\langle 0|$. Here we have ignored the ac Stark shift induced by classical fields, which is negligible in our regime (or it may be compensated by other lasers). Again, we assume $\delta^\mu \gg J^\mu \gg \omega^\mu$, which can be satisfied, along with the above condition, when

$$g^\mu/\Delta^\mu \gg J^\mu/g^\mu \gg \Omega^\mu/\Delta^\mu. \quad (6)$$

In this regime, the cavity photon is suppressed, and adiabatic elimination can be applied once more. We extend the method in Ref. [19] to keep up to the third-order terms and take only the subspace with no cavity photon. The effective Hamiltonian, in the rotating frame, can then be derived as

$$H = -t \sum_{p,q} [\sigma_{p+1,q}^+ \sigma_{p,q}^- e^{i(\theta_{p+1,q}^X - \theta_{p,q}^X)} + \sigma_{p,q+1}^+ \sigma_{p,q}^- e^{i(\theta_{p,q+1}^Y - \theta_{p,q}^Y)} + \text{H.c.}], \quad (7)$$

where the parameters are chosen such that $t = J^X(\omega^X/\delta^X)^2 = J^Y(\omega^Y/\delta^Y)^2$. It is easy to see that this Hamiltonian reduces to the Hamiltonian (2) if we adjust the phases of the classical fields as

$$\theta_{p,q}^X = \frac{2\pi}{\Phi_0} \int_{0,q}^{p,q} \mathbf{A}(\mathbf{r}) dl, \quad \theta_{p,q}^Y = \frac{2\pi}{\Phi_0} \int_{p,0}^{p,q} \mathbf{A}(\mathbf{r}) dl. \quad (8)$$

The FQH Hamiltonian (3) is obtained if we adjust these phases as $\theta_{p,q}^X = -pq2\pi\alpha$ and $\theta_{p,q}^Y = 0$. Note that the classical fields for $\theta_{p,q}^Y$ can be replaced by one global field.

In order to check the validity of the adiabatic approximation from Hamiltonian (5)–(7), we have performed a direct numerical diagonalization of Hamiltonian (5). We take a set of parameters $\delta^\mu/10 = 10\omega^\mu = J^\mu$, which corresponds to a case where $\Delta^\mu/1000 = g^\mu/100 = \Omega^\mu = J^\mu$. To eliminate the edge effects within a limited computational capability, we consider a periodic boundary condition (i.e., a torus). We consider a 4×4 lattice with $\alpha = 1/4$ and two bosons, hence four flux quanta in total, and the filling factor $\nu = 1/2$. In view of the fact that the cavity photon is suppressed, we restrict our calculation to the subspace wherein the maximum number of excitations in a cavity is limited to one, i.e., $\langle a_{p,q}^{X\dagger} a_{p,q}^X + a_{p,q}^{Y\dagger} a_{p,q}^Y + (|1\rangle\langle 1|)_{p,q}\rangle \leq 1$. Up to the modification due to the torus geometry and a different gauge [20], the ground state should be close to the Laughlin state (1) with $m = 2$. From our numerical diagonalization, the fidelity between the Laughlin state $|\Psi_2\rangle$ and the numerical ground state $|\Psi_G\rangle$ is found to be $F_G = |\langle\Psi_2|\Psi_G\rangle|^2 = 0.976$. We note that, when the ideal Hamiltonian (3) is diagonalized for the same 4×4 lattice, the fidelity of the ground state is found to be 0.989. The fidelity F_G converges to this value as δ^μ/J^μ and J^μ/ω^μ increase, which, however, demands more strong atom-cavity coupling. Note also that the non-unit fidelity is partly due to the finite α, which makes the effect of the lattice geometry non-negligible. The ground state fidelity of the Hamiltonian (3) increases close to 1 as α decreases [8].

In experiments, the ground state could be prepared by the adiabatic transformation, in a similar manner as in Ref. [8]. To show this, we consider the above 4×4 lattice system and deliberately add an energy shift $-\epsilon[(|1\rangle \times \langle 1|)_{0,0} + (|1\rangle\langle 1|)_{2,2}]$, which can be done in experiments by applying lasers at those sites to induce ac Stark shifts. When the energy shift ϵ is sufficiently large, the ground state is simply $|1\rangle_{0,0}|1\rangle_{2,2}$ with all other atoms in state $|0\rangle$. In Fig. 3, we plot the energy gap from the ground state to the nine lowest excited states with respect to the amount of the energy shift ϵ. The degeneracy of the ground state in the absence of the energy shift is due to the ambiguity of the center of mass in the torus geometry, which disappears in the plane geometry [20]. Aside from this degeneracy, the excited states have finite energy gaps which allow an adiabatic transformation. From this figure, it is apparent that the Laughlin ground state can be prepared by the following procedure: (i) Prepare the atoms in state $|1\rangle$ at sites chosen evenly in agreement with the filling factor ν, with all other atoms prepared in state $|0\rangle$. Initially all lasers are turned off. (ii) Apply lasers at the chosen sites to induce an ac Stark shift $-\epsilon|1\rangle\langle 1|$, with ϵ chosen moderately, e.g.,

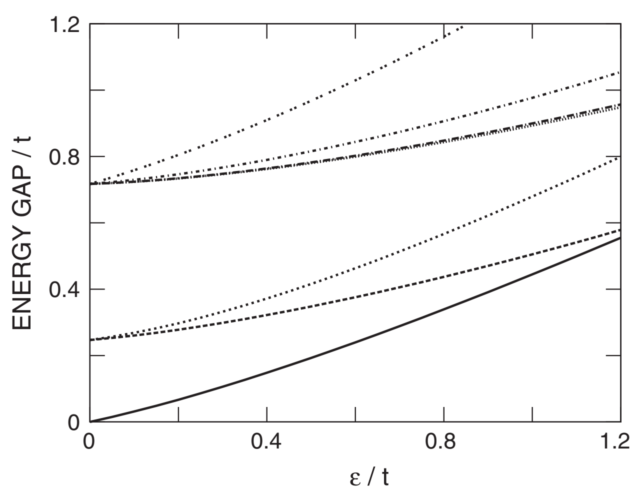

FIG. 3. Energy gap from the ground state to the nine lowest excited states for a 4×4 lattice in the periodic boundary condition in the presence of energy shift $-\epsilon|1\rangle\langle 1|$ applied at two sites chosen evenly. The filling factor is $\nu = 1/2$.

as the desired value of t. This energy shift does not change the atomic state. (iii) Gradually increase the Rabi frequencies Ω^X and Ω^Y to reach the desired value of t. (iv) Gradually decrease the energy shift ϵ to zero.

The quasiexcitation of the Laughlin state is generated when one magnetic flux quantum is adiabatically inserted through an infinitely thin solenoid piercing the 2D plane [3]. In the present system, we can choose the position of the quasiexcitation inside a lattice cell. Recalling that the vector potential outside a solenoid is given by $\vec{A}_s = \Phi_0/2\pi r\hat{\varphi}$, where r is the distance from the solenoid and $\hat{\varphi}$ is the azimuthal vector, the effect of the solenoid can be easily reflected in the phases of Eq. (8). Generation of the Laughlin state and the existence of the fractionally charged quasiexcitation (in the present case, fractionally excited bosons) could be examined by directly measuring the individual atoms: for example, by measuring the pair correlation functions [21]. Before the measurement, one may turn off all lasers so as to isolate the state from further evolution and decoherence.

Although the atomic excitation is highly suppressed, the atomic spontaneous decay is yet a prominent source of decoherence. If we denote by γ the spontaneous decay rate of an atom, the effective decay rate of the whole system due to the atomic decay is estimated as $N_b\gamma(\Omega/\Delta)^2$, where N_b denotes the total number of bosons in the system (we omit superscript X or Y for simplicity). On the other hand, the energy scale t in the Hamiltonian is given by $J(\Omega/g)^2$. In view of the condition $\Delta \gg g$, the former is still much smaller than the latter for moderate N_b if we assume $\gamma \lesssim J$. However, since the excitation gap is smaller than t, the attainable system size would be restricted in the experimental realization. Although the effective decay rate is decreased by increasing Δ, this in turn requires more stronger atom-cavity coupling rate g so as to satisfy the condition (6).

Finally, we stress the point that the asymmetric geometry introduced in Fig. 1 is not essential. That is, when the two orthogonal cavity modes have the same resonant frequency, one can also obtain the Hamiltonian (7) in the following way: We apply lasers with the same frequency, say, ω_1, in every second row so that they produce the spin exchange to the $\hat{x}$ direction, while applying lasers with a different frequency ω_2 in the other rows, which also produce the spin exchange to the $\hat{x}$ direction. If we choose those frequencies so that $|\omega_1 - \omega_2| \sim \delta^\mu$, they do not produce the spin exchange to the $\hat{y}$ direction. In the same manner, we apply lasers with frequencies ω_3 and ω_4 in every second column to produce the spin exchange to the $\hat{y}$ direction. By choosing those four frequencies to be sufficiently detuned, we can adjust the associated phases independently for each pair of the spin exchange.

We thank R. Fazio and B. Hessmo for helpful discussions. This work was supported by the Korea Research Foundation Grant No. KRF-2007-357-C00016 funded by the Korean Government (MOEHRD) and by the National Research Foundation & Ministry of Education, Singapore. S. B. thanks the Engineering and Physical Sciences Research Council (EPSRC) United Kingdom for financial support and for support through the Quantum Information Processing IRC (GR/S82176/01), the Royal Society, and the Wolfson Foundation.

[1] M. Lewenstein *et al.*, Adv. Phys. **56**, 243 (2007).
[2] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
[3] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[4] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).
[5] N. K. Wilkin and J. M. F. Gunn, Phys. Rev. Lett. **84**, 6 (2000); B. Paredes *et al.*, Phys. Rev. Lett. **87**, 010402 (2001).
[6] G. Juzeliūnas and P. Öhberg, Phys. Rev. Lett. **93**, 033602 (2004).
[7] D. Jaksch and P. Zoller, New J. Phys. **5**, 56 (2003).
[8] A. S. Sørensen, E. Demler, and M. D. Lukin, Phys. Rev. Lett. **94**, 086803 (2005).
[9] A. Stern, Phys. Rev. B **50**, 10 092 (1994).
[10] M. J. Hartmann, F. G. S. L. Brandão, and M. B. Plenio, Nature Phys. **2**, 849 (2006); D. G. Angelakis, M. F. Santos, and S. Bose, Phys. Rev. A **76**, 031805(R) (2007); A. D. Greentree *et al.*, Nature Phys. **2**, 856 (2006); D. G. Angelakis *et al.*, Phys. Lett. A **362**, 377 (2007).
[11] D. Rossini and R. Fazio, Phys. Rev. Lett. **99**, 186401 (2007); M. X. Huo, Y. Li, Z. Song, and C. P. Sun, Phys. Rev. A **77**, 022103 (2008); N. Na *et al.*, Phys. Rev. A **77**, 031803(R) (2008); M. Paternostro, G. S. Agarwal, and M. S. Kim, arXiv:0707.0846; D. G. Angelakis and A. Kay, New J. Phys. **10**, 023012 (2008).
[12] M. J. Hartmann, F. G. S. L. Brandão, and M. B. Plenio, Phys. Rev. Lett. **99**, 160501 (2007); J. Cho, D. G. Angelakis, and S. Bose, arXiv:0802.3365 [Phys. Rev. A (to be published)].
[13] K. Hennessy *et al.*, Nature (London) **445**, 896 (2007).
[14] J. Majer *et al.*, Nature (London) **449**, 443 (2007).
[15] T. Aoki *et al.*, Nature (London) **443**, 671 (2006).
[16] J. Cho, Phys. Rev. Lett. **99**, 020502 (2007).
[17] D. Jaksch *et al.*, Phys. Rev. Lett. **81**, 3108 (1998); L.-M. Duan, E. Demler, and M. D. Lukin, Phys. Rev. Lett. **91**, 090402 (2003).
[18] D. R. Hofstadter, Phys. Rev. B **14**, 2239 (1976).
[19] D. F. V. James and J. Jerke, Can. J. Phys. **85**, 625 (2007).
[20] F. D. M. Haldane, Phys. Rev. Lett. **55**, 2095 (1985); F. D. M. Haldane and E. H. Rezayi, Phys. Rev. B **31**, 2529 (1985).
[21] F. D. M. Haldane and E. H. Rezayi, Phys. Rev. Lett. **54**, 237 (1985).